Fuel-Optimal Direction-Cosine Attitude Control for Spin-Stabilized Axisymmetric Spacecraft

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IN this Note, the free-final-time, unbounded-control, fuel-optimal solutions are developed for a dynamic model which is based on a direction-cosine kinematic representation.[1] The optimal solutions are demonstrated to be either a) two impulses separated by a coasting arc or b) an impulse followed by a singular arc (depending on the initial conditions).

The state equations for the system are[1]†:

$$\begin{gathered}\dot{\omega}_1 + a\Omega\omega_2 = M_1/I = u(t)\cos\eta(t)/I \\ \dot{\omega}_2 - a\Omega\omega_1 = M_2/I = u(t)\sin\eta(t)/I \\ \omega_3(t) = \omega_3(0) = \Omega\end{gathered} \tag{1}$$

and

$$\begin{gathered}\dot{a}_{13} = \Omega a_{23} - \omega_2 a_{33},\ \dot{a}_{23} = -\Omega a_{13} + \omega_1 a_{33} \\ \dot{a}_{33} = \omega_2 a_{13} - \omega_1 a_{23}\end{gathered} \tag{2}$$

where ω_i = components of the angular velocity vector; $\Omega = \omega_3(t)$ = spin velocity; $a = (I_3 - I)/I$; I_i = moment of inertia for x_i axis; $I = I_1 = I_2$; and $u(t)$ and $\eta(t)$ are control magnitude and phase. The direction-cosine variables are not independent, since they satisfy the kinematic constraint

$$a_{13}^2 + a_{23}^2 + a_{33}^2 = 1 \tag{3}$$

These state equations define the attitude of a body-fixed x_i coordinate system relative to an inertial X_i system. The attitude-control problem is to place and maintain the body-fixed x_3 spin axis in alignment with the X_3 axis.

Necessary Conditions for Unbounded, Free-Time, Fuel-Optimal Control

The problem to be examined in this Note is as follows. Determine those control variables $u(t)$, $\eta(t)$ which transfer the state solutions from arbitrary initial conditions to the specified final state

$$\omega_1(T) = \omega_2(T) = a_{13}(T) = a_{23}(T) = 0,\ a_{33}(T) = 1 \tag{4}$$

while minimizing the fuel-performance index

$$J = \int_0^T Ku\,dt \tag{5}$$

In the above, the product Ku defines the rate of fuel consumption where $u \geq 0$, and the final time T is not specified. The Hamiltonian function for this system is defined as

$$\begin{aligned}\mathcal{H} = &-Ku + p_0(a_{13}^2 + a_{23}^2 + a_{33}^2 - 1) \\ &+ p_1(-a\Omega\omega_2 + u\cos\eta/I) + p_2(a\Omega\omega_1 + u\sin\eta/I) \\ &+ p_3(\Omega a_{23} - \omega_2 a_{33}) + p_4(-\Omega a_{13} + \omega_1 a_{33}) \\ &+ p_5(\omega_2 a_{13} - \omega_1 a_{23})\end{aligned} \tag{6}$$

The costate differential equations can then be expressed as

$$\begin{gathered}\dot{p} - ia\Omega p = i(a_{33}q - \alpha p_5),\ \dot{q} + i\Omega q - i\omega p_5 = 2p_0(t)\alpha \\ \dot{p}_5 - p_3\omega_2 + p_4\omega_1 = 2p_0(t)a_{33}\end{gathered} \tag{7}$$

where

$$\begin{gathered}p = p_1 + ip_2,\ q = p_3 + ip_4 \\ \omega = \omega_1 + i\omega_2 = |\omega|e^{i\theta},\ \alpha = a_{13} + ia_{23} = |\alpha|e^{i\phi}\end{gathered} \tag{8}$$

The maximum of the function $\mathcal{H}$ with respect to the control angle η is satisfied by

$$\sin\eta = p_2/|p|,\ \cos\eta = p_1/|p| \tag{9}$$

For unbounded u, it is necessary that

$$|p| \leq k,\ k = KI \tag{10}$$

If $|p| < k$, $u = 0$, whereas if $|p| = k$, the control u can be either unbounded (impulsive control) or bounded (singular or coasting arcs). The costate variables are required to be continuous, and the Hamiltonian function must satisfy

$$\mathcal{H}(t) = 0 \tag{11}$$

Extremal Fuel-Optimal Solutions

The nature of the extremal solutions for the system considered is most easily explained in terms of the properties of the motion of the system.[1] The free-motion solution for Eq. (1) is

$$\omega(t) = \omega^0 e^{ia\Omega t},\ \omega^0 = \omega(0) \tag{12}$$

where ω is defined in Eq. (8). In terms of the transformation

$$\alpha = ze^{i(\theta^0 + a\Omega t)} = (z_1 + iz_2)e^{i(\theta^0 + a\Omega t)} \tag{13}$$

the solution to Eq. (2) is

$$\begin{aligned}a_{33}(t) &= c\delta^0\,c\lambda^0 + s\delta^0\,s\lambda^0\,c\zeta \\ z(t) &= s\delta^0\,c\lambda^0 - c\delta^0\,s\lambda^0\,c\zeta + is\lambda^0\,s\zeta \\ \zeta &= \zeta^0 + \beta_0 t,\ \beta_0 = |H^0|/I\end{aligned} \tag{14}$$

where c ≡ cosine, s ≡ sine, and $|H^0|$ is the initial moment-of-momentum vector magnitude. Further, δ^0 and λ^0 are, respectively, the angles between H^0 and the x_3 and X_3 axes.

The same basic distinction between initial condition classes noted by Porcelli and Connolly[3] for the small-angle model holds here as well. Specifically, there are two basic optimal strategies depending on whether $\lambda^0 > \delta^0$ or $\lambda^0 < \delta^0$. The optimal strategy for $\lambda^0 < \delta^0$ is the simpler and consists of the two-impulse strategy:

$$u(t) = -J^1 e^{i(\theta^0 + a\Omega t^1)}\delta(t - t^1) - J(T)e^{i(\theta^1 + a\Omega T)}\delta(t - T) \tag{15}$$

where $\delta(\ )$ is the delta-dirac function; the initial firing time t^1 is arbitrary, and the phase of ω at t^1, θ^1, is defined by $\theta^1 = \theta^0 + a\Omega t^1$. The first impulse generates a new angular momentum vector H^1 such that the spin axis precesses into alignment with the X_3 axis, and the terminal impulse then reduces ω to zero when x_3 and X_3 are aligned. The costate solution which accompanies this strategy is given from Eqs. (6, 9, 10, and 15) by

$$p(t) = -ke^{i(\theta^0 + a\Omega t)},\ q(t) = p_5(t) = 0 \tag{16}$$

This large-angle optimal solution coincides with the small angle solution first formulated by Porcelli and Connolly.[3]

By contrast, there is a marked difference between the optimal solutions for the small-angle model[3,4] and the large-angle direction-cosine solution for $\lambda^0 > \delta^0$ initial conditions. Instead of a two-impulse solution, the optimum large-angle strategy consists of an impulse followed by a singular arc. This statement is confirmed by considering the special initial condition class $\delta^0 = 0$. The optimal control strategy consists of a forced nutation, and is accomplished by directing a constant-magnitude control torque towards X_3‡; i.e., Eq. (1)

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† The notation employed here coincides with that of Ref. 1.

‡ This is the customary[5] ideal control path which is taken in controlling passively damped spacecraft for which $\delta^0 = 0$.

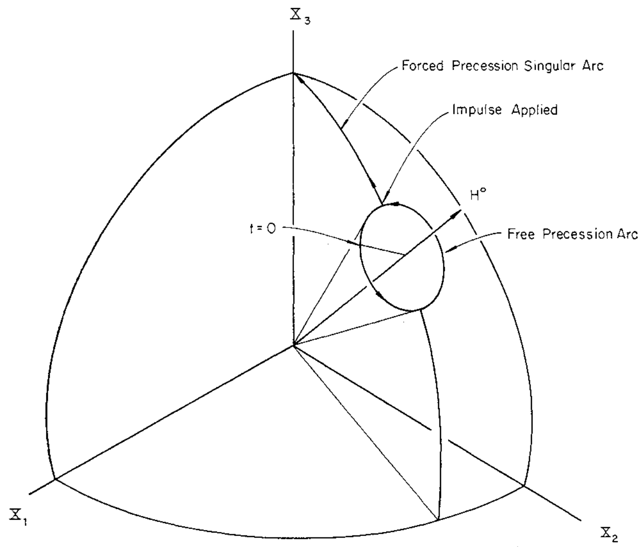

Fig. 1 Optimal state trajectory for $\lambda^0 > \delta^0$.

becomes

$$\dot{\omega} - ia\Omega\omega = Ue^{i\phi}/I \tag{17}$$

where ϕ is defined in Eq. (8), and U is a fixed control magnitude. The state solutions which correspond to Eq. (17) are

$$\omega(t) = i(U/I_3\Omega)e^{i\phi(t)}, \ \alpha(t) = \sin\gamma_3(t)e^{i\phi(t)}$$
$$a_{33}(t) = \cos\gamma_3(t) \tag{18}$$

where

$$\phi(t) = \phi^0 - \Omega t, \ \gamma_3(t) = \lambda^0 - (U/I_3\Omega)t \tag{19}$$

and the angle $\gamma_3(t)$ lies between the x_3 and X_3 axes. The costate solutions are

$$q(t) = -b\Omega k \cos\gamma_3 e^{i\phi}, \ p_5(t) = b\Omega k \sin\gamma_3$$
$$p(t) = ke^{i\phi}, \ b = a + 1 = I_3/I \tag{20}$$

The state and costate solutions given in Eqs. (18–20) yield an identical satisfaction of the necessary conditions except for the $\mathcal{H}(t) = 0$ condition given in Eq. (11), for which substitution yields

$$\mathcal{H}(t) = kU/I_3$$

Since the execution time T and control magnitude U are related by

$$U = I_3\Omega\gamma_3^0/T = I_3\Omega\lambda^0/T$$

Eq. (11) is satisfied for $T \to \infty$ and $U \to 0$.

The optimal control policy for general $\lambda^0 > \delta^0$ initial conditions (i.e., $\delta^0 \neq 0$) is illustrated in Fig. 1, and consists of a coasting arc from the initial state to the point

$$\gamma_3(t^1) = (\lambda^0 - \delta^0), \ \zeta(t^1) = 2\pi \tag{21}$$

At this time the impulse

$$J^1\delta(t - t^1) = -I|\omega^0|e^{i(\theta^0 + a\Omega t^1)} \tag{22}$$

is applied which eliminates ω, and places the spin axis on the singular arc to the origin.

The costate solution for the coast arc transfer is given by§

$$p(t) = k[-s^2\delta^0 - c^2\delta^0 \, c\zeta + i \, c\delta^0 \, s\zeta]e^{i(\theta^0 + a\Omega t)}$$
$$q(t) = kb\Omega[s\delta^0 \, s\lambda^0 + c\delta^0 \, c\lambda^0 \, c\zeta - i \, c\lambda^0 \, s\zeta]e^{i(\theta^0 + a\Omega t)} \tag{23}$$
$$p_5(t) = kb\Omega[s\lambda^0 \, c\delta^0 - c\lambda^0 \, s\delta^0 \, c\zeta]$$

§ Development of the general coast-arc costate solutions is given in the Appendix.

for $0 \le t \le t^1$. For $t^1 \le t \le T$ the solution follows from Eqs. (18–20).

Appendix: Coast-Arc Costate Solutions

An interesting fact about the formulated optimization problem is that the kinematic constraint given in Eq. (3) and included in the Hamiltonian equation (6) has no influence on the optimal solution. This fact is verified by considering the last two equations in Eqs. (7) (which are linear if $\omega(t)$ is specified). The complete solution to these equations is

$$q(t) = q^h(t) + A(t)\alpha(t)$$
$$p_5(t) = p_5{}^h(t) + A(t)a_{33}(t) \tag{A1}$$

where the superscript h denotes the homogeneous solution and the scalar function $A(t)$ is defined by $\dot{A}(t) = 2p_0(t)$. Substitution from Eq. (A1) into the first of Eqs. (7) yields

$$\dot{p} - ia\Omega p = i(a_{33}q^h - \alpha p_5{}^h)$$

Hence, the terms $2p_0\alpha$ and $2p_0a_{33}$ which arise in Eqs. (7) (due to the constraint) have no influence on the switching function. In addition, substitution from Eq. (A1) into Eq. (6) confirms that these terms have no influence on $\mathcal{H}(t)$ either, and the system whose solution is sought becomes, from Eq. (7),

$$\dot{p} - ia\Omega p = i(a_{33}q - \alpha p_5), \ \dot{q} + i\Omega q - i\omega p_5 = 0$$
$$\dot{p}_5 - p_3\omega_2 + p_4\omega_1 = 0 \tag{A2}$$

This result can be explained as follows. From Eq. (2), one has

$$2a_{13}\dot{a}_{13} + 2a_{23}\dot{a}_{23} + 2a_{33}\dot{a}_{33} = 0$$

Hence, if Eq. (3) is satisfied by initial conditions, it will continue to be satisfied for all time, since the constraints in Eq. (2) are included in Eq. (6).

The transformation

$$q(t) = ve^{i(\theta^0 + a\Omega t)} \tag{A3}$$

yields the solution for v and p_5

$$v/B = s\delta \, c\Lambda^0 - c\delta^0 \, s\Lambda^0 \, c\Psi + i \, s\Lambda^0 \, s\Psi$$
$$p_5/B = c\delta^0 \, c\Lambda^0 + s\delta^0 \, s\Lambda^0 \, c\Psi \tag{A4}$$

where B is a scalar constant, and $\Psi = \Psi^0 + \beta_0 t$. The solution for the first of Eqs. (A2) is

$$p(t) = p^0e^{ia\Omega t} + ie^{ia\Omega t}\int_0^t e^{-ia\Omega x}(a_{33}q - \alpha p_5)dx$$

From Eqs. (A3) and (13)

$$p(t) = p^0e^{ia\Omega t} + ie^{i(\theta^0 + a\Omega t)}\int_0^t (a_{33}v - zp_5)dx$$

Substituting from Eqs. (A4) and (14) followed by integration yields

$$p(t) = p^0e^{ia\Omega t} + ie^{i(\theta^0 + a\Omega t)}BZ(t)/\beta_0$$

where

$$Z(t) = -i \, c\delta^0 \, c\lambda^0 \, s\Lambda^0(c\Psi - c\Psi^0) + ic\delta^0 \, c\Lambda^0 \, s\lambda^0(c\zeta - c\zeta^0) - c\lambda^0 \, s\Lambda^0(s\Psi - s\Psi^0) + s\lambda^0 \, c\Lambda^0(s\zeta - s\zeta^0) + it \, s\lambda^0 \, s\delta^0 \, s\Lambda^0 \, s(\Psi^0 - \zeta^0)$$

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A Comparison between the Method of Integral Relations and the Method of Lines as Applied to the Blunt Body Problem

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THIS Note provides a direct comparison between the well-known method of integral relations[1,2] and the method of lines as applied to the problem of a blunt body with detached shock; the results tend to refute the notion that the method of integral relations is necessarily more accurate when only a few strips are employed.[1] The results of both methods agree favorably with each other and "exact" solutions over a wide range of Mach numbers even for the one-strip approximation used here. In some cases the method of lines even surpasses the method of integral relations in predicting certain aspects of the flow.

Both approaches retain the concept of dividing the subsonic shock layer into strips parallel to the body. The difference arises in transforming the original system of partial differential equations to one involving only ordinary differential equations. As opposed to the method of integral relations, which integrates the equations in the n direction (normal to the body) by assuming some appropriate polynomial expansion, the method of lines approximates the $(\partial/\partial n)(\;)$ terms directly through use of finite differences. In either case the variable values required at each interface become the dependent variables in a system of simultaneous ordinary differential equations in s, the arc length along the body. Physically speaking, the integral relations require that the conservation equations be satisfied in an average manner over the strips; with the method of lines they are satisfied exactly at each interface. Although only one strip was used in this study, the basic limitation on accuracy is the number of strips one is willing to program, since algebraic and computational difficulties rapidly multiply. Here the method of lines has a distinct advantage in that recursive relations can be derived, that allow the computer to perform much of the algebra as shown in Refs. 3 and 4.

In the following, the development as well as the notation closely follows that of Ref. 5, which treats the method of integral relations alone. The equations of motion for a compressible inviscid gas, the perfect gas law, and streamwise entropy conservation behind the shock are employed. Referring to Fig. 1, in body-oriented curvilinear coordinates (s,n) normalized to the body radius of curvature at its nose R_B one obtains the following relations in "divergence" form:

$$(\partial/\partial s)[r^j t] + (\partial/\partial n)[(1 + n/R)r^j h] = 0 \quad (1)$$

$$(\partial/\partial s)[r^j Z] + (\partial/\partial n)[(1 + n/R)r^j H] - G = 0 \quad (2)$$

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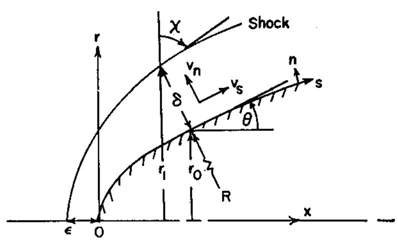

Fig. 1 Blunt body geometry.

where

$$t = \tau V_S,\ h = \tau V_n,\ Z = \rho V_S V_n$$
$$\tau = (1 - V^2)^{1/(\gamma-1)},\ H = \rho V_n^2 + kp,\ k = (\gamma - 1)/2\gamma$$
$$g = \rho V_S^2 + kp,\ G = (r^j/R)g + j(1 + n/R)kp\cos\theta$$

The velocities V are referred to the maximum adiabatic velocity, and the pressure and density, p and ρ, to their free-stream stagnation values. The parameter j is 0 or 1 for plane or axisymmetric flow, respectively.

The approximations are made as follows:

$$r^j t = r_0^j t_0 + (n/\delta)(r_1^j t_1 - r_0^j t_0)$$
$$r^j Z = (n/\delta) r_1^j Z_1,\ G = G_0 + (n/\delta)(G_1 - G_0)$$

$(\;)_0$ and $(\;)_1$ refer, respectively, to properties evaluated on the body and just behind the shock.

It should be noted here that with more than one strip, higher-order polynomials can be handled. Substituting the above in Eqs. (5) and (6) and taking

$$\int_0^\delta (\;)dn$$

(employing Leibnitz's rule), yields the method of integral relations. If instead one lets

$$\frac{\partial(\;)}{\partial n} = \frac{(\;)_1 - (\;)_0}{\delta},\ \frac{\partial(\;)}{\partial s} = \frac{1}{2}\left[\frac{d(\;)_1}{ds} + \frac{d(\;)_0}{ds}\right]$$

the method of lines is obtained. Through use of the oblique shock relations one can reduce the number of dependent variables to three, namely δ, χ, and V_{S_0}, where δ and χ are the local shock layer thickness and shock angle. The third equation is derived from the geometry of the shock. In either

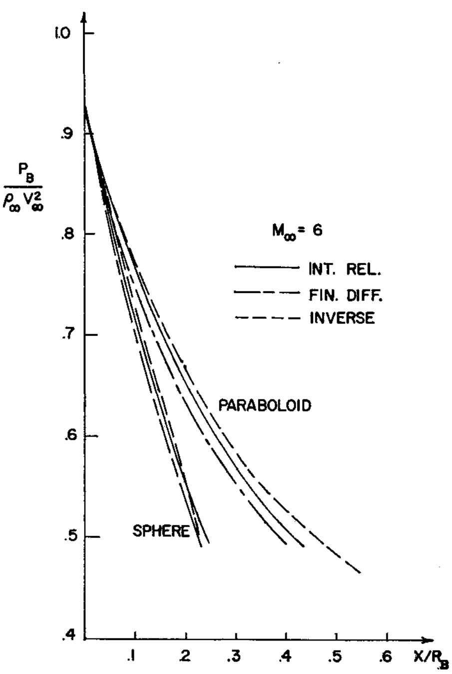

Fig. 2 Body pressure vs axial distance.